

THE FACE STRUCTURE AND GEOMETRY OF MARKED ORDER POLYHEDRA

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ABSTRACT. We study a class of polyhedra associated to marked posets. Examples of these polyhedra are Gelfand–Tsetlin polytopes and related polytopes that have appeared in the representation theory of semi-simple Lie algebras in the last decades. The faces of these polyhedra correspond to certain partitions of the underlying poset and we give a combinatorial characterization of these partitions. We specify a class of marked posets that give rise to polyhedra with facets in correspondence to the covering relations of the poset. On the convex geometrical side we describe the recession cone of the polyhedra, discuss products and give a Minkowski sum decomposition. We briefly discuss intersections with affine subspaces that have also appeared in representation theory and recently in the theory of finite Hilbert space frames.

1. INTRODUCTION

The mathematical developments leading to marked order polyhedra are split into two separate branches that just recently merged.

The first branch, started by Geissinger and Stanley in the 1980s, comes from order theory and combinatorial convex geometry. Given a finite poset P with a global maximum and a global minimum, Geissinger studied the polytope $\mathcal{O}(P)$ in \mathbb{R}^P consisting of order-preserving maps $P \rightarrow \mathbb{R}$ sending the minimum to 0 and the maximum to 1 in [8]. He found that vertices of this polytope correspond to non-trivial order ideals of P and more generally, that faces correspond to residually acyclic partitions of P . Geissinger also describes how the volume of $\mathcal{O}(P)$ is obtained from the number of linear extensions of P . These results reappear in [15], where Stanley called $\mathcal{O}(P)$ the *order polytope* associated to P and introduced a second polytope, the *chain polytope* $\mathcal{C}(P)$, with inequalities given by saturated chains in P . He introduced a piecewise linear *transfer map* $\mathcal{O}(P) \rightarrow \mathcal{C}(P)$ that yields an Ehrhart equivalence of the polytopes. In the same spirit of comparing these two polytopes associated to a finite poset, a group around Hibi and Li obtained results on unimodular equivalence and a bijection between edge sets in [12] and [13], respectively.

A second branch begins in the 1950s in representation theory of semi-simple complex Lie algebras, when Gelfand and Tsetlin introduced number patterns—soon attributed to them as *Gelfand–Tsetlin patterns*—to enumerate the elements in certain bases of irreducible representations in [9]. Given a fixed dominant integral weight λ for the general linear Lie algebra $\mathfrak{gl}_n(\mathbb{C})$, the corresponding irreducible representation $V(\lambda)$ has a basis enumerated by integral Gelfand–Tsetlin patterns. The defining conditions of these patterns give rise to the *Gelfand–Tsetlin polytope* associated to the dominant weight, so that the elements in the Gelfand–Tsetlin basis correspond to the lattice points in the Gelfand–Tsetlin polytope. In [10] and [2] the

authors used methods from enumerative combinatorics to study the number of vertices and the f -vector of Gelfand–Tsetlin polytopes, respectively.

Given another weight μ of the representation $V(\lambda)$, one can add certain linear conditions to the description of the Gelfand–Tsetlin polytope to obtain a different polytope, whose integral points enumerate a basis of the weight μ subspace of the irreducible representation $V(\lambda)$. These polytopes have been studied by De Loera and McAllister in [5], where they give a procedure to calculate the dimension of minimal faces corresponding to points in the polytope by using a method based on tiling matrices, very similar to the approach we take in this article. Similar techniques have been used in [1] to study integral points of Gelfand–Tsetlin polytopes.

These Gelfand–Tsetlin polytopes with additional linear conditions recently appeared the theory of finite Hilbert space frames in [4] as *polytopes of eigensteps*. In this setting they have been used to find parametrizations of frame varieties. In a special case the authors of [11] gave a non-redundant description of the polytope in terms of linear equations and inequalities, hence determining the dimension and number of facets of the polytope.

The branches started to merge in 2011, when Ardila, Bliem and Salazar generalized order and chain polytopes to *marked order and chain polytopes* in [3], allowing marking conditions other than just sending minima to 0 and maxima to 1. That is, given a finite poset P and a *marking* $\lambda: P^* \rightarrow \mathbb{R}$ of a subset $P^* \subseteq P$ containing all extremal elements, they defined the marked order polytope $\mathcal{O}(P, \lambda)$ to be the set of all order-preserving extensions of λ to all of P and generalized the marked chain polytope $\mathcal{C}(P, \lambda)$ accordingly. This generalization allowed them to consider Gelfand–Tsetlin polytopes and some other polytopes that appeared in representation theory as marked poset polytopes. They showed that the transfer map introduced by Stanley generalizes and still yields an Ehrhart equivalence $\mathcal{O}(P, \lambda) \rightarrow \mathcal{C}(P, \lambda)$ in the marked case. In [14] the authors study the number of lattice points in $\mathcal{O}(P, \lambda)$ when varying the values of λ . They find that this number is piecewise polynomial in the values of λ and use the generalized transfer map to obtain the same result for marked chain polytopes. They also characterize the faces of marked order polytopes by certain partitions of the underlying poset in the spirit of the original work by Geissinger and Stanley. However, the characterizations given in [14, Prop. 2.2, Prop. 2.3] are incorrect, since the mentioned conditions on the partitions are too weak. We state the correct characterization in Theorem 3.14. In [7] an attempt has been made to define a class of *regular* marked posets, where the facets of the associated marked order polytope are in correspondence with the covering relations of the posets, as is true in the unmarked case. However, the procedure in [7, Section 3] does not remove all redundant covering relations. We give a corrected definition of regular marked posets in Definition 3.18. Marked order and chain polytopes have been generalized in [6] to an Ehrhart equivalent family of *marked chain-order polytopes*, having marked order and marked chain polytopes as extremal cases.

In this article, we restrict our study to a potentially unbounded generalization of marked order polytopes. We start by defining marked posets and their associated marked order polyhedra in Section 2, describing different ways to look at the concept from an order theoretic, a convex geometric and a categorical point of view. We then study the face structure of marked order polyhedra in Section 3 and give a complete combinatorial characterization of partitions of the underlying poset corresponding to faces of the polyhedron. We specialize this characterization to

facets and show that regular marked posets have facets in correspondence with all covering relations of the poset. In Section 4, we focus on convex geometrical properties of marked order polyhedra. We describe the recession cone of the polyhedra, how disjoint unions of posets correspond to products of polyhedra and give a Minkowski sum decomposition. Furthermore, we show that marked order polyhedra with integral markings are always lattice polyhedra. We close by adding linear conditions to marked order polyhedra in Section 5, generalizing the result on dimensions of faces obtained in [5] to these conditional marked order polyhedra. Throughout the article we give examples to illuminate the obtained results.

2. MARKED POSETS AND THEIR ASSOCIATED POLYHEDRA

Definition 2.1. A *marked poset* (P, λ) is a finite poset P together with a subset $P^* \subseteq P$ of *marked elements* and an order-preserving *marking* $\lambda: P^* \rightarrow \mathbb{R}$. The marking λ is called *strict* if $\lambda(a) < \lambda(b)$ whenever $a < b$. A map $f: (P, \lambda) \rightarrow (P', \lambda')$ between marked posets is an order-preserving map $f: P \rightarrow P'$ such that $f(P^*) \subseteq (P')^*$ and $\lambda'(f(a)) = \lambda(a)$ for all $a \in P^*$.

When talking about a poset P we will always denote its partial order by \leq and covering relations by \prec . That is, for $p, q \in P$ we write $p \prec q$ to indicate that $p < q$ and $p \leq s \leq q$ implies $s = p$ or $s = q$.

To study marked posets and the polyhedra we will associate to them, it is sometimes useful to take a more categorically minded point of view on marked posets. From the definition above we see that marked posets form a category MPos . Letting Pos denote the category of posets and order-preserving maps, we can describe MPos as a category of certain diagrams in Pos . A marked poset (P, λ) is a diagram

$$P \hookrightarrow P^* \xrightarrow{\lambda} \mathbb{R}$$

in Pos , where $P^* \hookrightarrow P$ is the inclusion of an induced subposet P^* in a finite poset P . A map $f: (P, \lambda) \rightarrow (P', \lambda')$ is a commutative diagram

$$\begin{array}{ccc} P & \hookrightarrow & P^* \xrightarrow{\lambda} \mathbb{R} \\ \downarrow f & & \downarrow f|_{P^*} \quad \parallel \\ P' & \hookrightarrow & P'^* \xrightarrow{\lambda'} \mathbb{R} \end{array}$$

To each marked poset (P, λ) we associate a polyhedron $\mathcal{O}(P, \lambda)$ in \mathbb{R}^P .

Definition 2.2. Let (P, λ) be a marked poset. The *marked order polyhedron* $\mathcal{O}(P, \lambda)$ associated to (P, λ) is the set of all $x \in \mathbb{R}^P$ such that $x_p \leq x_q$ for all $p, q \in P$ with $p \leq q$ and $x_a = \lambda(a)$ for all $a \in P^*$. When P^* contains all extremal elements of P , the polyhedron $\mathcal{O}(P, \lambda)$ is bounded. In this case we say $\mathcal{O}(P, \lambda)$ is the *marked order polytope* associated to (P, λ) .

In more geometric terms, this definition is equivalent to

$$\mathcal{O}(P, \lambda) = \bigcap_{p < q} H_{p < q}^+ \cap \bigcap_{a \in P^*} H_a,$$

where $H_{p < q}^+$ is the half-space in \mathbb{R}^P defined by $x_p \leq x_q$ and H_a is the hyperplane defined by $x_a = \lambda(a)$.

An interval $[a, b]$ in a marked poset (P, λ) is called *constant* if $a, b \in P^*$ and $\lambda(a) = \lambda(b)$. In this case $x_p = \lambda(a)$ for all $x \in \mathcal{O}(P, \lambda)$ and $p \in [a, b]$. With

this terminology, a marking λ is strict if and only if (P, λ) contains no non-trivial constant intervals.

We can also think of the marked order polyhedron $\mathcal{O}(P, \lambda)$ as the set of all extensions of λ to order-preserving maps $x: P \rightarrow \mathbb{R}$ with $x|_{P^*} = \lambda$. That is, the set of all poset maps $x: P \rightarrow \mathbb{R}$ such that the diagram

$$\begin{array}{ccccc} P & \longleftrightarrow & P^* & \xrightarrow{\lambda} & \mathbb{R} \\ & & \searrow x & \nearrow & \\ & & & & \end{array}$$

commutes. Putting together the diagram of a map $f: (P, \lambda) \rightarrow (P', \lambda')$ between marked posets and that of a point $x \in \mathcal{O}(P', \lambda')$, we see that we obtain a point $f^*(x)$ in $\mathcal{O}(P, \lambda)$ given by $f^*(x) = x \circ f$:

$$\begin{array}{ccccc} P & \longleftrightarrow & P^* & \xrightarrow{\lambda} & \mathbb{R} \\ \downarrow f & & \downarrow f|_{P^*} & & \parallel \\ P' & \longleftrightarrow & P'^* & \xrightarrow{\lambda'} & \mathbb{R} \\ & & \searrow x & \nearrow & \\ & & & & \end{array}$$

Hence, letting Polyh denote the category of polyhedra and affine maps, we have a contravariant functor $\mathcal{O}: \text{MPos} \rightarrow \text{Polyh}$ sending a marked poset (P, λ) to the marked order polyhedron $\mathcal{O}(P, \lambda)$ and a map f between marked posets to the induced map f^* described above.

As we will see in the next proposition, any marking λ can be extended to P and any strict marking can be extended to a strictly order-preserving map $P \rightarrow \mathbb{R}$.

Proposition 2.3. *Let (P, λ) be a marked poset. The associated marked order polyhedron is non-empty and if λ is strict, there is a point $x \in \mathcal{O}(P, \lambda)$ such that $x_p < x_q$ whenever $p < q$.*

Proof. The order on \mathbb{R} is dense and unbounded. Hence, whenever $a < c$ in \mathbb{R} there is a $b \in \mathbb{R}$ such that $a < b < c$ and for any $b \in \mathbb{R}$ there are $a, c \in \mathbb{R}$ such that $a < b < c$. Since P is finite this allows us to successively extend λ to an order-preserving map on P . In fact, we can find an order-preserving extension x of λ such that for $p < q$ we have $x_p = x_q$ if and only if there are $a, b \in P^*$ such that $a \leq p, q \leq b$ and $\lambda(a) = \lambda(b)$. In particular, when λ was strict we can always find a strictly order-preserving extension. \square

Example 2.4. We consider the marked order polytope given by the marked poset (P, λ) in Figure 1. The blue labels name elements in P , while the red labels in boxes correspond to values of the elements of P^* under the marking λ . Since all coordinates apart from x_p and x_q are fixed, we can project $\mathcal{O}(P, \lambda)$ from \mathbb{R}^P to $\mathbb{R}^{\{p, q\}} \cong \mathbb{R}^2$ and obtain the 2-dimensional picture in Figure 1.

3. THE FACE STRUCTURE AND FACETS OF MARKED ORDER POLYHEDRA

In this section, we study the face structure of $\mathcal{O}(P, \lambda)$. As it turns out, the faces of marked order polyhedra correspond to certain partitions of the underlying poset P . Our goal is to characterize those partitions combinatorially. We associate to each point x in $\mathcal{O}(P, \lambda)$ a partition π_x of P , that will suffice to describe the minimal face of $\mathcal{O}(P, \lambda)$ containing x . The partitions that are obtained in this way from points of the polyhedron will then—ordered by refinement—capture the polyhedron's face structure.

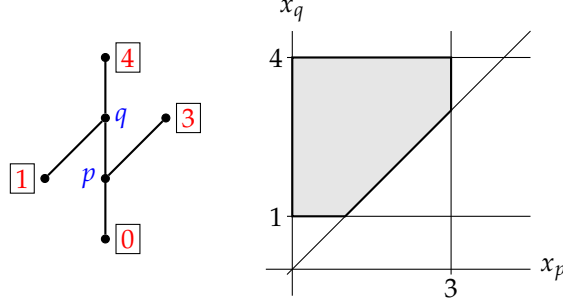


FIGURE 1. The marked poset (P, λ) from Example 2.4 and the associated marked order polytope $\mathcal{O}(P, \lambda)$.

Definition 3.1. Let $Q = \mathcal{O}(P, \lambda)$ be a marked order polyhedron. To each $x \in Q$ we associate a partition π_x of P induced by the transitive closure of the relation

$$p \sim_x q \quad \text{if} \quad x_p = x_q \text{ and } p, q \text{ are comparable.}$$

We may think of π as being obtained by first partitioning P into blocks of constant values under x and then splitting those blocks into connected components with respect to the Hasse diagram of P .

Given any partition π of P , we call a block $B \in \pi$ *free* if $P^* \cap B = \emptyset$ and denote by $\tilde{\pi}$ the set of all free blocks of π . Note that any $x \in \mathcal{O}(P, \lambda)$ is constant on the blocks of π_x and the values on the non-free blocks of π_x are determined by λ .

Let $x \in Q$ be a point of a polyhedron. We denote the minimal face of Q containing x by F_x . Hence, F_x is the unique face having x in its relative interior. Equivalently, F_x is the intersection of all faces of Q containing x .

Proposition 3.2. Let $x \in Q = \mathcal{O}(P, \lambda)$ be a point of a marked order polyhedron with associated partition $\pi = \pi_x$. We have

$$F_x = \left\{ y \in Q \mid y \text{ is constant on the blocks of } \pi \right\}$$

and $\dim F_x = |\tilde{\pi}|$.

Proof. For $p < q$ in P let $H_{p < q} = \partial H_{p < q}^+$ be the hyperplane defined by $x_p = x_q$ in \mathbb{R}^P . The minimal face of a point $x \in Q$ is then given by

$$F_x = Q \cap \bigcap_{\substack{p < q, \\ x_p = x_q}} H_{p < q}.$$

A point $y \in Q$ satisfies $y_p = y_q$ for all $p < q$ with $x_p = x_q$ if and only if y is constant on the blocks of π_x . Thus, F_x is indeed given by all $y \in Q$ constant on the blocks of π_x .

To determine the dimension of F_x , we consider its affine hull $\text{aff}(F_x)$. It is obtained by intersecting the affine hull of Q with all $H_{p < q}$ such that $x_p = x_q$. The affine hull of Q itself is the intersection of all H_a for $a \in P^*$ and all $H_{p < q}$ such that $y_p = y_q$ for all $y \in Q$. Putting these facts together, we have

$$\text{aff}(F_x) = \bigcap_{a \in P^*} H_a \cap \bigcap_{\substack{p < q, \\ y_p = y_q \forall y \in Q}} H_{p < q} \cap \bigcap_{\substack{p < q, \\ x_p = x_q}} H_{p < q} = \bigcap_{a \in P^*} H_a \cap \bigcap_{\substack{p < q, \\ x_p = x_q}} H_{p < q}.$$

This is exactly the set of all y constant on the blocks of π_x and satisfying $y_a = \lambda(a)$ for all $a \in P^*$. Such y are uniquely determined by values on the free blocks of π_x and $\dim(F_x) = |\widetilde{\pi}_x|$ as desired. \square

Corollary 3.3. *If λ is a strict marking on P , the dimension of $\mathcal{O}(P, \lambda)$ is equal to the number of unmarked elements in P .*

Proof. Since all coordinates in P^* are fixed by λ , we always have $\dim \mathcal{O}(P, \lambda) \leq |P \setminus P^*|$. If λ is strict, there is a point $x \in \mathcal{O}(P, \lambda)$ such that $x_p < x_q$ whenever $p < q$ by Proposition 2.3. Hence, π_x is the partition of P into singletons and $\dim F_x = |\widetilde{\pi}_x| = |P \setminus P^*|$. We conclude that $F_x = \mathcal{O}(P, \lambda)$, so x is a relative interior point and the marked order polyhedron has the desired dimension. \square

Corollary 3.4. *Let $x \in Q = \mathcal{O}(P, \lambda)$ be a point of a marked order polyhedron. For $y \in Q$ we have $y \in F_x$ if and only if π_x is a refinement of π_y .*

Proof. By Proposition 3.2, $y \in F_x$ if and only if y is constant on the blocks of π_x . Let y be constant on the blocks of π_x . Any block B of π_x is connected with respect to the Hasse diagram of P and y takes constant values on B , hence B is contained in a block of π_y by construction and π_x is a refinement of π_y . Now let $y \in Q$ with π_x being a refinement of π_y . We conclude that y is constant on the blocks of π_x , since it is constant on the blocks of π_y and π_x is a refinement of π_y . \square

Corollary 3.5. *Given any two points $x, y \in \mathcal{O}(P, \lambda)$, we have $F_y \subseteq F_x$ if and only if π_x is a refinement of π_y . In particular $F_y = F_x$ if and only if $\pi_y = \pi_x$.* \square

Hence, the partition of $\mathcal{O}(P, \lambda)$ into relative interiors of its faces is the same as the partition given by $x \sim y$ if $\pi_x = \pi_y$ and we can associate to each non-empty face F a partition π_F with $\pi_F = \pi_x$ for any x in the relative interior of F . We call a partition π of P a *face partition* of (P, λ) if $\pi = \pi_F$ for some non-empty face of $\mathcal{O}(P, \lambda)$. We arrive at the following description of face lattices of marked order polyhedra.

Corollary 3.6. *Let $Q = \mathcal{O}(P, \lambda)$ be a marked order polyhedron. The poset $\mathcal{F}(Q) \setminus \{\emptyset\}$ of non-empty faces of Q is isomorphic to the induced subposet of the partition lattice on P given by all face partitions of (P, λ) .* \square

For the marked order polytope from Example 2.4, we illustrated the face partitions in Figure 2. The free blocks are highlighted in blue round shapes, non-free blocks in red angular shapes. We see that the dimensions of the faces are given by the numbers of free blocks in the associated face partitions and that face inclusions correspond to refinements of partitions.

In order to characterize the face partitions of a marked poset (P, λ) combinatorially, we introduce some properties of partitions of P .

Definition 3.7. Let (P, λ) be a marked poset. A partition π of P is *connected* if the blocks of π are connected as induced subposets of P . It is *P -compatible*, if the relation \leq defined on π as the transitive closure of

$$B \leq C \quad \text{if} \quad p \leq q \text{ for some } p \in B, q \in C$$

is antisymmetric. In this case \leq is a partial order on π . A P -compatible partition π is called *(P, λ) -compatible*, if whenever $a \in B \cap P^*$ and $b \in C \cap P^*$ for some blocks $B \leq C$, we have $\lambda(a) \leq \lambda(b)$.

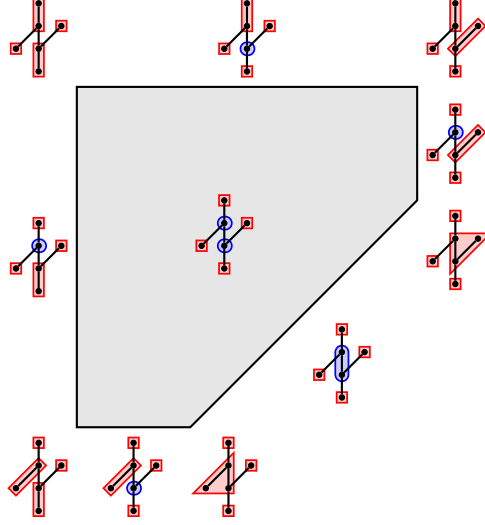


FIGURE 2. The face partitions of the marked order polytope in Example 2.4.

Remark 3.8. Whenever a partition π of a poset P is P -compatible, it is also *convex*. That is, for $a < b < c$ with a and c in the same block $B \in \pi$, we also have $b \in B$, since otherwise the blocks containing a and b would cause the relation on π to be antisymmetric. This implies that the blocks in a connected, P -compatible partition are not just connected as induced subposets of P but even connected as induced subgraphs of the Hasse diagram of P .

Proposition 3.9. Let (P, λ) be a marked poset. A (P, λ) -compatible partition π of P gives rise to a marked poset $(P/\pi, \lambda/\pi)$ where P/π is the poset of blocks in π , $(P/\pi)^* = \pi \setminus \tilde{\pi}$ and $\lambda/\pi: (P/\pi)^* \rightarrow \mathbb{R}$ is defined by $(\lambda/\pi)(B) = \lambda(a)$ for any $a \in B \cap P^*$. Furthermore, the quotient map $P \rightarrow P/\pi$ defines a map $(P, \lambda) \rightarrow (P/\pi, \lambda/\pi)$ of marked posets.

Proof. Since π is P -compatible, the blocks of π form a poset P/π as in Definition 3.7. Since π is (P, λ) -compatible, we have $\lambda(a) = \lambda(b)$ whenever $a, b \in B \cap P^*$ for some non-free block $B \in \pi$. Hence, the map λ/π is well-defined. It is order-preserving by the definition of (P, λ) -compatibility. Furthermore, we have a commutative diagram

$$\begin{array}{ccccc} P & \longleftarrow & P^* & \xrightarrow{\lambda} & \mathbb{R} \\ \downarrow & & \downarrow & & \parallel \\ P/\pi & \longleftarrow & (P/\pi)^* & \xrightarrow{\lambda/\pi} & \mathbb{R}. \end{array}$$

Thus, we have a quotient map $(P, \lambda) \rightarrow (P/\pi, \lambda/\pi)$. □

Proposition 3.10. Every face partition π_F of (P, λ) is (P, λ) -compatible, connected and the induced marking on $(P/\pi_F, \lambda/\pi_F)$ is strict.

Proof. Let F be a non-empty face of $\mathcal{O}(P, \lambda)$. It is obvious that π_F is connected by construction, since it is given by the transitive closure of a relation that only relates pairs of comparable elements. To verify that π_F is P -compatible, we need to check that the induced relation \leq on the blocks of π_F is antisymmetric. Assume we have blocks $B, C \in \pi_F$ such that $B \leq C$ and $C \leq B$. Since $B \leq C$, there is a finite

sequence of blocks $B = X_1, X_2, \dots, X_k, X_{k+1} = C$ such that for $i = 1, \dots, k$ there are some $p_i \in X_i, q_i \in X_{i+1}$ with $p_i \leq q_i$. Take any x in the relative interior of F , then $x_{p_i} \leq x_{q_i}$ for $i = 1, \dots, k$ and since x is constant on the blocks of π_F , we have $x_{q_i} = x_{p_{i+1}}$ for $i = 1, \dots, k-1$. To summarize, we have

$$x_{p_1} \leq x_{q_1} = x_{p_2} \leq x_{q_2} = \dots \leq \dots = x_{p_k} \leq x_{q_k}. \quad (1)$$

Hence, the constant value x takes on B is less than or equal to the constant value x takes on C . Since we also have $C \leq B$, we conclude that x takes equal values on the blocks B and C . From (1) we conclude that x takes equal values on all blocks X_i . From the definition of $\pi_x = \pi_F$ it follows that the blocks X_i are in fact all equal, in particular $B = C$ and the relation is antisymmetric.

To see that π_F is (P, λ) -compatible, let $B, C \in \pi$ be non-free blocks with $B \leq C$. By the same argument as above, we know that any $x \in F$ has constant value on B less than or equal to the constant value on C , so $\lambda(a) \leq \lambda(b)$ for marked $a \in B, b \in C$. If $\lambda(a) = \lambda(b)$ we have $B = C$, again by the same argument as above, so the induced marking is strict. \square

Given any partition π of P , we can define a polyhedron F_π contained in $\mathcal{O}(P, \lambda)$ by

$$F_\pi = \left\{ y \in Q \mid y \text{ is constant on the blocks of } \pi \right\}.$$

If $\pi = \pi_F$ is a face partition of (P, λ) , we have $F_\pi = F$ by Proposition 3.2. However, F_π is not a face for all partitions π of P .

As long as π is (P, λ) -compatible, we can show that the polyhedron F_π is affinely isomorphic to the marked order polyhedron $\mathcal{O}(P/\pi, \lambda/\pi)$. The isomorphism will be induced by the quotient map $P \rightarrow P/\pi$. Our first step is to verify that this induced map is indeed an injection.

Lemma 3.11. *Let $f: (P, \lambda) \rightarrow (P', \lambda')$ be a map of marked posets. If f is surjective, the induced map $f^*: \mathcal{O}(P', \lambda') \rightarrow \mathcal{O}(P, \lambda)$ is injective.*

Proof. Let $x, y \in \mathcal{O}(P', \lambda')$ such that $f^*(x) = f^*(y)$. Given any $p \in P'$ we need to show $x_p = y_p$. Since f is surjective, $p = f(q)$ for some $q \in P$ and thus

$$x_p = x_{f(q)} = f^*(x)_q = f^*(y)_q = y_{f(q)} = y_p. \quad \square$$

Proposition 3.12. *Let (P, λ) be a marked poset and π a (P, λ) -compatible partition. The quotient map $q: (P, \lambda) \rightarrow (P/\pi, \lambda/\pi)$ induces an injection*

$$q^*: \mathcal{O}(P/\pi, \lambda/\pi) \hookrightarrow \mathcal{O}(P, \lambda)$$

with image $q^(\mathcal{O}(P/\pi, \lambda/\pi)) = F_\pi$.*

Proof. By Lemma 3.11 we know that q^* is an injection. Hence, we only need to verify that F_π is the image of q^* . The image is contained in F_π , since whenever p and p' are in the same block $B \in \pi$, we have

$$q^*(x)_p = x_{q(p)} = x_B = x_{q(p')} = q^*(x)_{p'}.$$

Hence, all $q^*(x)$ are constant on the blocks of π . Conversely, given any point $y \in \mathcal{O}(P, \lambda)$ constant on the blocks of π , we obtain a well defined map $x: P/\pi \rightarrow \mathbb{R}$ sending each block to the constant value y_p for all p in the block. This map is a point $x \in \mathcal{O}(P/\pi, \lambda/\pi)$ mapped to y by q^* . \square

The previous proposition tells us, that whenever we have a (P, λ) -compatible partition π , the marked order polyhedron $\mathcal{O}(P/\pi, \lambda/\pi)$ is affinely isomorphic to the polyhedron $F_\pi \subseteq \mathcal{O}(P, \lambda)$ via the embedding q^* induced by the quotient map. From now on, we refer to affine isomorphisms arising this way as the *canonical affine isomorphism* $\mathcal{O}(P/\pi, \lambda/\pi) \cong F_\pi$.

Corollary 3.13. *For every non-empty face F of a marked order polyhedron $\mathcal{O}(P, \lambda)$ we have a canonical affine isomorphism $\mathcal{O}(P/\pi_F, \lambda/\pi_F) \cong F$.*

We are now ready to state and prove the characterization of face partitions of marked posets.

Theorem 3.14. *A partition π of a marked poset (P, λ) is a face partition if and only if it is (P, λ) -compatible, connected and the induced marking on $(P/\pi, \lambda/\pi)$ is strict.*

Proof. The fact that face partitions satisfy the above properties is the statement of Proposition 3.10. Now let π be a partition of P that is (P, λ) -compatible, connected and induces a strict marking λ/π . By Proposition 2.3, there is a point $z \in \mathcal{O}(P/\pi, \lambda/\pi)$ such that $z_B < z_C$ whenever $B < C$. Let $x \in \mathbb{R}^P$ be the point in the polyhedron $F_\pi \subseteq \mathcal{O}(P, \lambda)$ obtained as the image of z under the canonical affine isomorphism $\mathcal{O}(P/\pi, \lambda/\pi) \xrightarrow{\sim} F_\pi$. We claim that $\pi = \pi_x$, so π is a face partition. Since x is constant on the blocks of π and π is connected, we know that π is a refinement of π_x . Now assume that the equivalence relation \sim_x defining π_x relates elements in different blocks of π . In this case, there are blocks $B \neq C$ of π with elements $p \in B, q \in C$ such that $x_p = x_q$ and $p < q$. This implies that $z_B = z_C$ and $B < C$, a contradiction to the choice of z . Hence $\pi = \pi_x$ and π is a face partition of (P, λ) . \square

Remark 3.15. To decide whether a given partition π of a marked poset (P, λ) satisfies the conditions in Theorem 3.14, it is enough to know the linear order on $\lambda(P^*)$. The exact values of the marking are irrelevant. Hence, the face lattice of $\mathcal{O}(P, \lambda)$ is determined solely by discrete, combinatorial data. In fact, since the directions of facet normals do not depend on the values of λ , we can conclude that the normal fan $\mathcal{N}(\mathcal{O}(P, \lambda))$ is determined by this combinatorial data. However, the affine isomorphism type of $\mathcal{O}(P, \lambda)$ does depend on the exact values of λ .

Example 3.16. We construct a continuous family $(Q_t)_{t \in [0,1]}$ of marked order polytopes, whose underlying marked posets all yield the same combinatorial data in the sense of Remark 3.15, but Q_s and Q_t are affinely isomorphic if and only if $s = t$. Let (P, λ_t) be the marked poset shown in Figure 3. Letting t vary in $[0, 1]$, we obtain for each t a different affine isomorphism type, since 2 of the vertices of Q_t will move, while the other 3 stay fixed and are affinely independent. However, all Q_t share the same normal fan and are in particular combinatorially equivalent.

We continue our study of the face structure of marked order polyhedra by having a closer look at facets. Since inequalities in the description of marked order polyhedra come from covering relations in the underlying poset, we expect a correspondence of facets to certain covering relations. If the marked poset satisfies a certain regularity condition, the facets are indeed in bijection to the covering relation. Hence, if we can change the underlying poset of a marked order polyhedron to a regular one, without changing the associated polyhedron, we obtain an enumeration of facets. We start by modifying an arbitrary marked poset to a strict one by contracting constant intervals.

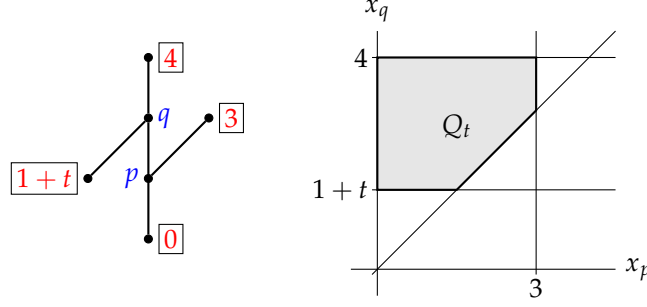


FIGURE 3. The marked poset (P, λ_t) from Example 3.16 and the associated marked order polytope $Q_t = \mathcal{O}(P, \lambda_t)$.

Proposition 3.17. *Given any marked poset (P, λ) , the partition π induced by the relations $a \sim p$ and $p \sim b$ whenever $[a, b]$ is a constant interval containing p yields a strictly marked poset $(P/\pi, \lambda/\pi)$ such that $\mathcal{O}(P/\pi, \lambda/\pi) \cong F_\pi = \mathcal{O}(P, \lambda)$ via the canonical affine isomorphism.*

Proof. Let $x \in \mathcal{O}(P, \lambda)$ be a point constructed as in the proof of Proposition 2.3. By construction we have $x_p = x_q$ for $p < q$ if and only if there are $a, b \in P^*$ with $a \leq p < q \leq b$ with $\lambda(a) = \lambda(b)$. Thus, we conclude that $\pi_x = \pi$ and π is a face partition of $\mathcal{O}(P, \lambda)$. Since every point of $\mathcal{O}(P, \lambda)$ satisfies $x_a = x_p = x_b$ whenever $[a, b]$ is a constant interval containing p , we conclude that F_π is indeed the whole polyhedron. Hence $\mathcal{O}(P/\pi, \lambda/\pi) \cong F_\pi = \mathcal{O}(P, \lambda)$, where λ/π is a strict marking by Proposition 3.10. \square

Definition 3.18. Let (P, λ) be a marked poset. A covering relation $p < q$ is called *non-redundant*, if for all marked elements a, b satisfying $a \leq q$ and $p \leq b$, we have $a = b$ or $\lambda(a) < \lambda(b)$. Otherwise the covering relation is called *redundant*. The marked poset (P, λ) is called *regular*, if all its covering relations are non-redundant.

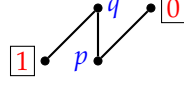
Apart from the desired correspondence of covering relations and facets, regularity of marked posets implies some useful properties of the marked poset itself.

Proposition 3.19. *Let (P, λ) be a regular marked poset. The following conditions are satisfied:*

- i) *the marking λ is strict,*
- ii) *there are no covering relations between marked elements,*
- iii) *every element in P covers and is covered by at most one marked element.*

Proof. i) When $a < b$ are marked elements of P , there is some covering relation $p < q$ such that $a \leq p < q \leq b$. Since $a \leq q$ and $p \leq b$, we have $\lambda(a) < \lambda(b)$ by regularity.
 ii) When $b < a$ is a covering relation between marked elements, we have $\lambda(a) < \lambda(b)$ by choosing $p = b, q = a$ in the regularity condition. This is a contradiction to λ being order-preserving.
 iii) When $a, b < q$ for marked a, b , the regularity condition for $a \leq q$ and $b \leq b$ implies $a = b$ or $\lambda(a) < \lambda(b)$. By the same argument we get $a = b$ or $\lambda(b) < \lambda(a)$. We conclude that $a = b$. \square

Remark 3.20. The conditions in Proposition 3.19 are necessary, but not sufficient for (P, λ) to be regular. The marked poset



satisfies all three conditions, but the covering relation $p \prec q$ is redundant.

Theorem 3.21. *Let (P, λ) be a regular marked poset. The facets of $\mathcal{O}(P, \lambda)$ correspond to the covering relations in (P, λ) .*

Proof. Since (P, λ) is strictly marked, the dimension of $\mathcal{O}(P, \lambda)$ is equal to the number of unmarked elements in P . Hence, a facet F corresponds to a (P, λ) -compatible, connected partition π of P such that λ/π is strict and π has exactly $|P \setminus P^*| - 1$ free blocks. We claim that the number of non-free blocks of π is $|P^*|$. Assume there are marked elements $a \neq b$ in a common block B of π . Since π has $|P \setminus P^*| - 1$ free blocks, at most one unmarked element can be in a non-free block. Since (P, λ) is regular, there are no covering relations between marked elements. Hence, since B is connected as an induced subgraph of the Hasse diagram of P and contains both a and b , it also contains the only unmarked element p in a non-free block, and we have one of the following four situations: $a \prec p \prec b$, $a \succ p \succ b$, $a \prec p \succ b$ or $a \succ p \prec b$. Since a and b are in the same block, they are identically marked and the first two possibilities contradict λ being strict. The other two possibilities contradict regularity, since p covers—or is covered by—more than one marked element. Hence, π has exactly $|P^*|$ non-free blocks and we conclude that π has $|P| - 1$ blocks overall. Therefore, π consists of $|P| - 2$ singletons and a single connected 2-element block corresponding to a covering relation of P .

Conversely, let $p \prec q$ be a covering relation of P . We claim that the partition π with the only non-singleton block $\{p, q\}$ is a face partition with $|P \setminus P^*| - 1$ free blocks. Since (P, λ) is regular, it contains no covering relation between marked elements and π has exactly $|P \setminus P^*| - 1$ free blocks. Since $\{p, q\}$ is the only non-singleton block and $p \prec q$, the partition π is connected and P -compatible. To verify that π is (P, λ) -compatible and λ/π is strict, let B, C be non-free blocks of π with $a \in B \cap P^*$ and $b \in C \cap P^*$ such that $B \leq C$. When $B = C$, we have $a = b$ and $\lambda(a) = \lambda(b)$. When $B < C$, we conclude $a < b$ or $a \leq q, p \leq b$, since $\{p, q\}$ is the only non-trivial block. In both cases, regularity implies $\lambda(a) < \lambda(b)$. \square

Now that we established a regularity condition on marked posets that guarantees a bijection between covering relations in P and facets of the marked order polyhedron, we explain how to transform any given marked poset to a regular one.

Proposition 3.22. *Let (P, λ) be a strictly marked poset. Redundant covering relations in P can be successively removed without changing the associated marked order polyhedron. This yields a regular marked poset (P', λ) defining the same marked order polyhedron.*

Proof. Let $p \prec q$ be a redundant covering relation in P . That is, there are marked elements $a \neq b$ satisfying $a \leq q, p \leq b$ and $\lambda(a) \geq \lambda(b)$. Let P' be obtained from P by removing the covering relation $p \prec q$ from P . Obviously $\mathcal{O}(P, \lambda)$ is contained in $\mathcal{O}(P', \lambda)$.

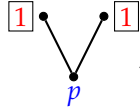
Now let $x \in \mathcal{O}(P', \lambda)$. To verify that x is a point of $\mathcal{O}(P, \lambda)$, we have to show $x_p \leq x_q$. Since λ is a strict marking on P , we can not have $a \leq p$. Otherwise $a \leq p \leq b$ implies $a < b$, in contradiction to $\lambda(a) \geq \lambda(b)$. Hence, removing the

covering relation $p \prec q$ we still have $a \leq' q$ in P' . By the same argument $p \leq' b$. Thus, by the defining conditions of $\mathcal{O}(P', \lambda)$, we have

$$x_p \leq x_b = \lambda(b) \leq \lambda(a) = x_a \leq x_q.$$

Therefore $x \in \mathcal{O}(P, \lambda)$ and we conclude $\mathcal{O}(P', \lambda) = \mathcal{O}(P, \lambda)$. This process can be repeated until all redundant covering relations have been removed, resulting in a regular marked poset defining the same marked order polyhedron. \square

Remark 3.23. Note that Proposition 3.22 does not imply, that all covering relations that are redundant in (P, λ) can be removed simultaneously. Removing a single redundant covering relation can lead to other redundant covering-relations becoming non-redundant. In the marked poset



both covering relations are redundant. However, removing any of the two covering relations renders the remaining covering relation non-redundant.

Given any marked poset (P, λ) , we can apply the constructions of Proposition 3.17 and Proposition 3.22 to obtain a regular marked poset (P', λ') defining the same marked order polyhedron up to canonical affine isomorphism.

4. CONVEX GEOMETRY OF MARKED ORDER POLYHEDRA

In this section we study some convex geometric properties of marked order polyhedra. We describe recession cones, a correspondence between disjoint unions of posets and products of polyhedra, characterize pointedness and use these results to obtain a Minkowski sum decomposition. At the end of the section we show that marked posets with integral markings always give rise to lattice polyhedra.

Proposition 4.1. *The recession cone of $\mathcal{O}(P, \lambda)$ is $\mathcal{O}(P, 0)$, where $0: P^* \rightarrow \mathbb{R}$ is the zero marking on the same domain as λ .*

Proof. By definition of the recession cone, we have

$$\text{rec } \mathcal{O}(P, \lambda) = \left\{ y \in \mathbb{R}^P \mid x + ty \in \mathcal{O}(P, \lambda) \text{ for all } x \in \mathcal{O}(P, \lambda), t \geq 0 \right\}.$$

Let $x \in \mathcal{O}(P, \lambda)$, $y \in \mathcal{O}(P, 0)$ and $t \geq 0$. Since $y_p \leq y_q$ implies $ty_p \leq ty_q$ and $y_a = 0$ implies $ty_a = 0$, we have $ty \in \mathcal{O}(P, 0)$ and may just assume $t = 1$. Furthermore, whenever $x_p \leq x_q$ and $y_p \leq y_q$, we have $x_p + y_p \leq x_q + y_p$ and whenever $x_a = \lambda(a)$ and $y_a = 0$ we have $x_a + y_a = \lambda(a)$. Hence, $x + y \in \mathcal{O}(P, \lambda)$ and $\mathcal{O}(P, 0) \subseteq \text{rec } \mathcal{O}(P, \lambda)$.

Now fix $y \in \text{rec } \mathcal{O}(P, \lambda)$ and let $x \in \mathcal{O}(P, \lambda)$. We have $x + y \in \mathcal{O}(P, \lambda)$ so for all $a \in P^*$ we have $x_a + y_a = \lambda(a)$, so $y_a = 0$. Furthermore, for $p \leq q$ we have $x_p \leq x_q$ and $x_p + ty_p \leq x_q + ty_q$ for all $t \geq 0$. Hence, for all $t > 0$, we have

$$y_p \leq \frac{x_q - x_p}{t} + y_q,$$

where $x_q - x_p \geq 0$. This implies $y_p \leq y_q$ and thus $y \in \mathcal{O}(P, 0)$. \square

Proposition 4.2. *Let (P_1, λ_1) and (P_2, λ_2) be marked posets on disjoint sets. Let the marking $\lambda_1 \cup \lambda_2: P_1^* \cup P_2^* \rightarrow \mathbb{R}$ on $P_1 \cup P_2$ be given by λ_1 on P_1^* and λ_2 on P_2^* . The marked order polyhedron $\mathcal{O}(P_1 \cup P_2, \lambda_1 \cup \lambda_2)$ is equal to the product $\mathcal{O}(P_1, \lambda_1) \times \mathcal{O}(P_2, \lambda_2)$ under the canonical identification $\mathbb{R}^{P_1 \cup P_2} = \mathbb{R}^{P_1} \times \mathbb{R}^{P_2}$.*

Proof. The defining equations and inequalities of a product polyhedron $Q_1 \times Q_2$ in $\mathbb{R}^{P_1} \times \mathbb{R}^{P_2}$ are obtained by imposing both the defining conditions of Q_1 and Q_2 . In case of $Q_1 = \mathcal{O}(P_1, \lambda_1)$ and $Q_2 = \mathcal{O}(P_2, \lambda_2)$ these are exactly the defining conditions of $\mathcal{O}(P_1 \cup P_2, \lambda_1 \cup \lambda_2)$. \square

Note that this relation between disjoint unions of marked posets and products of the associated marked order polyhedra may be expressed as the contravariant functor $\mathcal{O}: \text{MPos} \rightarrow \text{Polyh}$ sending coproducts to products.

We now characterize marked posets whose associated polyhedra are *pointed*. A pointed polyhedron is one that has at least one vertex, or equivalently does not contain a line. The importance of pointedness lies in the fact that pointed polyhedra are determined by their vertices and recession cone. To be precise, a pointed polyhedron is the Minkowski sum of its recession cone and the polytope obtained as the convex hull of its vertices.

Proposition 4.3. *A marked order polyhedron $\mathcal{O}(P, \lambda)$ is pointed if and only if each connected component of P contains a marked element.*

Proof. Let P_1, \dots, P_k be the connected components of P with $\lambda_i = \lambda|_{P_i}$ the restricted markings. By inductively applying Proposition 4.2, we have a decomposition

$$\mathcal{O}(P, \lambda) = \mathcal{O}(P_1, \lambda_1) \times \dots \times \mathcal{O}(P_k, \lambda_k).$$

Hence, $\mathcal{O}(P, \lambda)$ is pointed if and only if each $\mathcal{O}(P_i, \lambda_i)$ is pointed, reducing the statement to the case of P being connected.

Let (P, λ) be a connected marked poset and suppose $v \in \mathcal{O}(P, \lambda)$ is a vertex. By Proposition 3.2 the corresponding partition π has no free blocks. Hence, either P is empty or it has at least as many marked elements as the number of blocks in π .

Conversely, if P is connected and contains marked elements, the following procedure yields a vertex v of $\mathcal{O}(P, \lambda)$: start by setting $v_a = \lambda(a)$ for all $a \in P^*$. Pick any $p \in P$ such that v_p is not already determined and p is adjacent to some q in the Hasse-diagram of P with v_q already determined. Set v_p to be the maximum of all determined v_q with p covering q or the minimum of all determined v_q with p covered by q . Continue until all v_p are determined.

In each step, the defining conditions of $\mathcal{O}(P, \lambda)$ are respected and the procedure determines all v_p since P is connected and contains a marked element. By construction, each block of π_v will contain a marked element and thus v is a vertex by Proposition 3.2. \square

Proposition 4.4. *Given two markings $\lambda_1, \lambda_2: P^* \rightarrow \mathbb{R}$ on the same poset P , the Minkowski sum $\mathcal{O}(P, \lambda_1) + \mathcal{O}(P, \lambda_2)$ is contained in $\mathcal{O}(P, \lambda_1 + \lambda_2)$, where $\lambda_1 + \lambda_2$ is the marking sending $a \in P^*$ to $\lambda_1(a) + \lambda_2(a)$.*

Proof. Let $x \in \mathcal{O}(P, \lambda_1)$ and $y \in \mathcal{O}(P, \lambda_2)$. For any relation $p \leq q$ in P we have $x_p \leq x_q$ and $y_p \leq y_q$, hence $x_p + y_p \leq x_q + y_q$. For $a \in P^*$ we have $x_a + y_a = \lambda_1(a) + \lambda_2(a) = (\lambda_1 + \lambda_2)(a)$. Thus, $x + y \in \mathcal{O}(P, \lambda_1 + \lambda_2)$. \square

We are now ready to give a Minkowski sum decomposition of marked order polyhedra, such that the marked posets associated to the summands have 0-1-markings. The decomposition is a generalization of [16, Theorem 4] and [14, Corollary 2.10], where the bounded case with P^* being a chain in P is considered.

Theorem 4.5. *Let (P, λ) be a marked poset with $P^* \neq \emptyset$ and $\lambda(P^*) = \{c_0, c_1, \dots, c_k\}$ with $c_0 < c_1 < \dots < c_k$. Let $c_{-1} = 0$ and define markings $\lambda_i: P^* \rightarrow \mathbb{R}$ for $i = 0, \dots, k$ by*

$$\lambda_i(a) = \begin{cases} 0, & \text{if } \lambda(a) < c_i, \\ 1, & \text{if } \lambda(a) \geq c_i. \end{cases}$$

Then $\mathcal{O}(P, \lambda)$ decomposes as the weighted Minkowski sum

$$\mathcal{O}(P, \lambda) = \sum_{i=0}^k (c_i - c_{i-1}) \mathcal{O}(P, \lambda_i).$$

Proof. Since

$$\lambda = c_0 \lambda_0 + (c_1 - c_0) \lambda_1 + \dots + (c_k - c_{k-1}) \lambda_k$$

and in general $\mathcal{O}(P, c\lambda) = c \mathcal{O}(P, \lambda)$, one inclusion follows immediately from Proposition 4.4. For the other inclusion, first assume that $\mathcal{O}(P, \lambda)$ is pointed. In this case, it is enough to consider vertices and the recession cone. Since the underlying posets and sets of marked elements agree for all polytopes in consideration, they all have the same recession cone $\mathcal{O}(P, 0)$ by Proposition 4.1. Let $v \in \mathcal{O}(P, \lambda)$ be a vertex. The associated face partition π has no free blocks and on each block v takes some constant value in $\lambda(P^*)$. For fixed $i \in \{0, \dots, k\}$ we enumerate the blocks of π where v takes constant value c_i by $B_{i,1}, \dots, B_{i,r_i}$. For a block $B \in \pi$ denote by $w_B = \sum_{p \in B} e_p \in \mathbb{R}^P$ the labeling of P with all entries in B equal to 1, all other entries equal to 0. This yields a description of v as

$$v = \sum_{i=0}^k c_i \sum_{j=1}^{r_i} w_{B_{i,j}}.$$

For $i = 0, \dots, k$ define points $v^{(i)} \in \mathbb{R}^P$ by

$$v^{(i)} = (c_i - c_{i-1}) \sum_{l=i}^k \sum_{j=1}^{r_l} w_{B_{l,j}}.$$

This gives a decomposition of v as $v^{(0)} + \dots + v^{(k)}$. It remains to be checked that each $v^{(i)}$ is a point in the corresponding Minkowski summand. Since $v^{(0)}$ is just constant c_0 on the whole poset and λ_0 is the marking of all ones, we have $v^{(0)} \in c_0 \mathcal{O}(P, \lambda_0)$. Fix $i \in \{1, \dots, k\}$. For $p \leq q$ we have $v_p \leq v_q$ and thus $p \in B_{i,j}$, $q \in B_{i',j'}$ for $i \leq i'$ by the chosen enumeration of blocks. Hence, by definition of $v^{(i)}$, the inequality $v_p^{(i)} \leq v_q^{(i)}$ is equivalent to one of the three inequalities $0 \leq 0$, $0 \leq c_i - c_{i-1}$ or $c_i - c_{i-1} \leq c_i - c_{i-1}$, all being true. The marking conditions of $\mathcal{O}(P, (c_i - c_{i-1}) \lambda_i)$ are satisfied by $v^{(i)}$ as well, so $v^{(i)} \in (c_i - c_{i-1}) \mathcal{O}(P, \lambda_i)$. We conclude that

$$v = \sum_{i=0}^k v^{(i)} \in \sum_{i=0}^k (c_i - c_{i-1}) \mathcal{O}(P, \lambda_i)$$

for each vertex v of $\mathcal{O}(P, \lambda)$. Hence, the proof is finished for the case of $\mathcal{O}(P, \lambda)$ being pointed.

When $\mathcal{O}(P, \lambda)$ is not pointed, we can decompose $P = P' \cup P''$ where P' consists of all connected components without marked elements and P'' consists of all other components. Letting λ' and λ'' be the respective restrictions of λ , we have $\mathcal{O}(P, \lambda) = \mathcal{O}(P', \lambda') \times \mathcal{O}(P'', \lambda'')$ by Proposition 4.2, where $\mathcal{O}(P', \lambda')$ is not pointed

while $\mathcal{O}(P'', \lambda'')$ is, by Proposition 4.3. Applying the previous result to $\mathcal{O}(P'', \lambda'')$ we obtain

$$\mathcal{O}(P, \lambda) = \mathcal{O}(P', \lambda') \times \left(\sum_{i=0}^k (c_i - c_{i-1}) \mathcal{O}(P'', \lambda_i'') \right).$$

Since P' contains no marked elements, it is equal to its recession cone and we have

$$\mathcal{O}(P', \lambda') = \sum_{i=0}^k \mathcal{O}(P', \lambda').$$

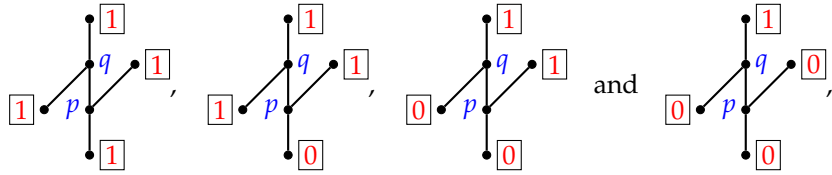
Therefore, using the identity $\sum_{i=0}^k P_i \times \sum_{i=0}^k Q_i = \sum_{i=0}^k (P_i \times Q_i)$ for products of Minkowski sums, we obtain

$$\begin{aligned} \mathcal{O}(P, \lambda) &= \left(\sum_{i=0}^k \mathcal{O}(P', \lambda') \right) \times \left(\sum_{i=0}^k (c_i - c_{i-1}) \mathcal{O}(P'', \lambda_i'') \right) \\ &= \sum_{i=0}^k (\mathcal{O}(P', \lambda') \times \mathcal{O}(P'', (c_i - c_{i-1}) \lambda_i'')) \\ &= \sum_{i=0}^k \mathcal{O}(P' \cup P'', \lambda' \cup (c_i - c_{i-1}) \lambda_i''). \end{aligned}$$

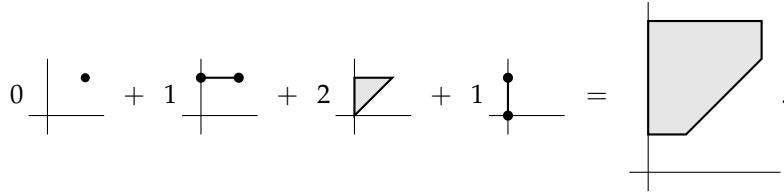
Since P' did not contain any markings that could be affected by scaling, the factors $(c_i - c_{i-1})$ can be put as dilation factors in front of the polyhedra. Again, since P' is unmarked, we have $\lambda' \cup \lambda_i'' = \lambda_i$ and $P' \cup P'' = P$, so we obtain the desired Minkowski sum decomposition. \square

Remark 4.6. When $\mathcal{O}(P, \lambda)$ is a polytope, $\mathcal{O}(P, 1)$ is just a point and the marked poset polytopes $\mathcal{O}(P, \lambda_i)$ appearing in the Minkowski sum decomposition of Theorem 4.5 may all be expressed as ordinary poset polytopes as discussed by Stanley [15] and Geissinger [8] by contracting constant intervals and dropping redundant conditions.

Example 4.7. We apply the Minkowski sum decomposition of Theorem 4.5 to the marked order polytope (P, λ) from Example 2.4. Since $\lambda(P^*) = \{0, 1, 3, 4\}$ in this example, we obtain the four new markings $\lambda_0, \lambda_1, \lambda_2$ and λ_3 given by



respectively. The associated marked order polytopes and their weighted Minkowski sum are



We finish by considering marked posets with integral markings. An important class of polytopes are *lattice polytopes*. These are polytopes where all vertices have integral coordinates. When all markings on a poset P are integral and the extrema

are all contained in P^* , we will see that $\mathcal{O}(P, \lambda)$ is indeed a lattice polytope. To make a more general statement that still holds in the unbounded case, we need to generalize this notion to polyhedra.

Definition 4.8. A *lattice polyhedron* $Q \subseteq \mathbb{R}^n$ is a polyhedron that can be expressed as a Minkowski sum of a lattice polytope and a rational polyhedral cone.

A simple fact about lattice polyhedra we will need is that products of lattice polyhedra are lattice polyhedra. This is an immediate consequence of the Minkowski sum identity $(Q + R) \times (Q' + R') = (Q \times Q') + (R \times R')$ we already used, together with products of lattice polytopes being lattice polytopes and products of rational cones being rational cones.

Proposition 4.9. Let (P, λ) be a marked poset such that $\lambda(P^*) \subseteq \mathbb{Z}$. Then the marked order polyhedron $\mathcal{O}(P, \lambda)$ is a lattice polyhedron.

Proof. When $\mathcal{O}(P, \lambda)$ is pointed, it is enough to show that all vertices are lattice points and the recession cone is rational. By Proposition 3.2 the face partitions associated to vertices have no free blocks. Hence, all coordinates are contained in $\lambda(P^*)$, so vertices are lattice points. The recession cone is obtained as $\mathcal{O}(P, 0)$ by Proposition 4.1, which is a rational polyhedral cone.

If $\mathcal{O}(P, \lambda)$ is not pointed, we use the decomposition $P = P' \cup P''$ of P into unmarked connected components in P' and the other components in P'' . As in the previous proof, we obtain a product decomposition $\mathcal{O}(P, \lambda) = \mathcal{O}(P', \lambda') \times \mathcal{O}(P'', \lambda'')$ by Proposition 4.2. Since P'^* is empty we know that $\mathcal{O}(P', \lambda')$ is a rational polyhedral cone. Since all connected components of P'' contain marked elements, we know that $\mathcal{O}(P'', \lambda'')$ is pointed and hence a lattice polyhedron by the previous argument. We conclude that $\mathcal{O}(P, \lambda)$ is a lattice polyhedron. \square

5. CONDITIONAL MARKED ORDER POLYHEDRA

In this section we study intersections of marked order polyhedra with affine subspaces. We describe an affine subspace U of \mathbb{R}^P by a linear map $s: \mathbb{R}^P \rightarrow \mathbb{R}^k$ and a vector $b \in \mathbb{R}^k$, such that $U = s^{-1}(b)$. Hence, U is the space of solutions to the linear system $s(x) = b$.

Definition 5.1. Given a marked poset (P, λ) , a linear map $s: \mathbb{R}^P \rightarrow \mathbb{R}^k$ and $b \in \mathbb{R}^k$, we define the *conditional marked order polyhedron* $\mathcal{O}(P, \lambda, s, b)$ as the intersection $\mathcal{O}(P, \lambda) \cap s^{-1}(b)$.

The faces of $\mathcal{O}(P, \lambda, s, b)$ correspond to the faces of $\mathcal{O}(P, \lambda)$ whose relative interior meets $s^{-1}(b)$. Hence, they are also given by face partitions. However, given a face partition π of $\mathcal{O}(P, \lambda)$, deciding whether it is a face partition of $\mathcal{O}(P, \lambda, s, b)$ can not be done combinatorially in general. The problem is in determining whether the linear system $s(x) = b$ admits a solution in the relative interior of F_π . We come back to this issue later in the section. Still, given a point $x \in \mathcal{O}(P, \lambda, s, b)$, we obtain a face partition π_x and we can find the dimension of $F_x \subseteq \mathcal{O}(P, \lambda, s, b)$ by calculating a kernel of a linear map associated to π_x .

Given a partition π of P , we define the linear injection $r_\pi: \mathbb{R}^{\tilde{\pi}} \rightarrow \mathbb{R}^P$ by

$$r_\pi(z)_p = \begin{cases} z_B & \text{if } p \text{ is an element of the free block } B \in \tilde{\pi}, \\ 0 & \text{otherwise.} \end{cases}$$

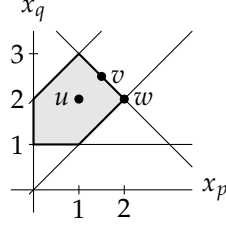


FIGURE 4. The conditional marked order polytope $\mathcal{O}(P, \lambda, s, b)$ from Example 5.4 together with three points on faces of different dimensions.

We can describe r_π as taking a labeling z of the free blocks of π with real numbers and making it into a labeling of P with real numbers, by putting the values given by z on elements in free blocks, while labeling elements in non-free blocks with zero. If π is a face partition of $\mathcal{O}(P, \lambda)$, we have seen in the proof of Proposition 3.2 that the affine hull of $F_\pi \subseteq \mathcal{O}(P, \lambda)$ is a translation of $\text{im}(r_\pi)$. The following proposition is a generalization of this observation to conditional marked order polyhedra.

Proposition 5.2. *Let x be a point of $\mathcal{O}(P, \lambda, s, b)$ with associated face partition $\pi = \pi_x$. Let U be the linear subspace of \mathbb{R}^P parallel to the affine hull of the face $F_x \subseteq \mathcal{O}(P, \lambda, s, b)$. The map r_π restricts to an isomorphism $\ker(s \circ r_\pi) \xrightarrow{\sim} U$. In particular $\dim F_x = \dim \ker(s \circ r_\pi)$.*

Proof. Let F'_x be the minimal face of $\mathcal{O}(P, \lambda)$ containing x , so that $F_x = F'_x \cap s^{-1}(b)$. For the affine hulls we also have $\text{aff}(F_x) = \text{aff}(F'_x) \cap s^{-1}(b)$. Letting U' be the linear subspace parallel to $\text{aff}(F'_x)$, just as U is the linear subspace parallel to $\text{aff}(F_x)$, we obtain

$$U = U' \cap \ker(s) = \text{im}(r_\pi) \cap \ker(s),$$

since $\ker(s)$ is the linear subspace parallel to $s^{-1}(b)$. This description implies that r_π restricts to an isomorphism $\ker(s \circ r_\pi) \xrightarrow{\sim} U$. \square

Remark 5.3. In the special case of Gelfand–Tsetlin polytopes with linear conditions given by a weight μ , this result appeared in [5] in terms of tiling matrices associated to points in the polytope. The tiling matrix is exactly the matrix associated to the linear map $s \circ r_\pi$ in the setting of Gelfand–Tsetlin patterns.

Example 5.4. Let (P, λ) be the linear marked poset

$$\boxed{0} \prec p \prec q \prec r \prec s \prec \boxed{5}$$

and impose the linear conditions $x_p + x_r = 4$ and $x_q + x_s = 6$ on $\mathcal{O}(P, \lambda)$. We describe these conditions by intersecting with $s^{-1}(b)$ for the linear map $s: \mathbb{R}^P \rightarrow \mathbb{R}^2$ given by $s(x) = (x_p + x_r, x_q + x_s)$ and $b = (4, 6)$. Any point in $\mathcal{O}(P, \lambda, s, b)$ is determined by x_p and x_q , so we can picture the polytope in \mathbb{R}^2 . Expressing the five inequalities in terms of x_p, x_q using the linear conditions, we obtain

$$0 \leq x_p, \quad x_p \leq x_q, \quad x_q \leq 4 - x_p, \quad x_q \leq 2 + x_p, \quad 1 \leq x_q.$$

The resulting polytope in $\mathbb{R}^{\{p, q\}} \cong \mathbb{R}^2$ is illustrated in Figure 4.

We want to calculate the dimensions of the minimal faces of $\mathcal{O}(P, \lambda, s, b)$ containing the points $u = (1, 2)$, $v = (1.5, 2.5)$ and $w = (2, 2)$ in \mathbb{R}^2 . In \mathbb{R}^P these points and their associated partitions of P are

$$0 \mid 1 \mid 2 \mid 3 \mid 4 \mid 5, \quad 0 \mid 1.5 \mid 2.5 \mid 2.5 \mid 3.5 \mid 5, \quad \text{and} \quad 0 \mid 2 \mid 2 \mid 2 \mid 4 \mid 5.$$

Hence, we have 4, 3 and 2 free blocks, respectively. The associated linear maps $s \circ r_\pi$ can be represented by the matrices

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \text{ and } \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix},$$

respectively. The kernels of these maps have dimension 2, 1 and 0 corresponding to the dimensions of the minimal faces containing u , v and w as one can see in Figure 4.

Given any (P, λ) -compatible partition of P , we obtained a polyhedron F'_π contained in $\mathcal{O}(P, \lambda)$ in the previous section. Hence, we have a polyhedron F_π contained in $\mathcal{O}(P, \lambda, s, b)$ given by $F_\pi = F'_\pi \cap s^{-1}(b)$. As in the unconditional case, these polyhedra are canonically affine isomorphic to conditional marked order polyhedra given by the quotient $(P/\pi, \lambda/\pi)$.

Proposition 5.5. *Let (P, λ) be a marked poset, π a (P, λ) -compatible partition and $s: \mathbb{R}^P \rightarrow \mathbb{R}^k$, $b \in \mathbb{R}^k$ be given. Define s/π to be the composition $s \circ q^*$, where q^* is the inclusion $\mathbb{R}^{P/\pi} \hookrightarrow \mathbb{R}^P$ induced by the quotient map of marked posets. The polyhedron $F_\pi \subseteq \mathcal{O}(P, \lambda, s, b)$ is affinely isomorphic to the conditional marked order polyhedron $\mathcal{O}(P/\pi, \lambda/\pi, s/\pi, b)$ via the canonical isomorphism obtained by restricting q^* .*

Proof. By definition, F_π is the intersection of the face F'_π of $\mathcal{O}(P, \lambda)$ with $s^{-1}(b)$. We know that q^* restricts to an affine isomorphism $\mathcal{O}(P/\pi, \lambda/\pi) \xrightarrow{\sim} F'_\pi$. Hence, F_π is contained in the image of q^* as well and we have

$$F_\pi = F'_\pi \cap s^{-1}(b) = F'_\pi \cap \text{im } q^* \cap s^{-1}(b) = F'_\pi \cap q^*((s \circ q^*)^{-1}(b)).$$

We may write F'_π as $q^*(\mathcal{O}(P/\pi, \lambda/\pi))$ and use injectivity of q^* to obtain

$$F_\pi = q^*(\mathcal{O}(P/\pi, \lambda/\pi)) \cap q^*((s/\pi)^{-1}(b)) = q^*(\mathcal{O}(P/\pi, \lambda/\pi) \cap (s/\pi)^{-1}(b)).$$

By definition of conditional marked order polyhedra, this is just the injective image of $\mathcal{O}(P/\pi, \lambda/\pi, s/\pi, b)$ under q^* , which finishes the proof. \square

When F is a non-empty face of $\mathcal{O}(P/\pi, \lambda/\pi, s/\pi, b)$ we have an associated partition $\pi = \pi_F$, so that $F = F_\pi$. Thus, we obtain the same corollary on faces of conditional marked order polyhedra as in the unconditional case.

Corollary 5.6. *For every non-empty face F of a conditional marked order polyhedron $\mathcal{O}(P, \lambda, s, b)$ we have a canonical affine isomorphism $\mathcal{O}(P/\pi_F, \lambda/\pi_F, s/\pi_F, b) \cong F$. \square*

The next proposition will allow us to consider *any* polyhedron as a conditional marked order polyhedron up to affine isomorphism. Thus, there is little hope to understand general conditional marked order polyhedra any better than we understand polyhedra in general.

Proposition 5.7. *Every polyhedron is affinely isomorphic to a conditional marked order polyhedron.*

Proof. Let $Q \subseteq \mathbb{R}^n$ be a polyhedron given by linear equations and inequalities

$$\begin{aligned} \sum_{i=1}^n a_{ki} x_i &= c_k & \text{for } k = 1, \dots, s, \\ \sum_{i=1}^n b_{li} x_i &\leq d_l & \text{for } l = 1, \dots, t. \end{aligned}$$

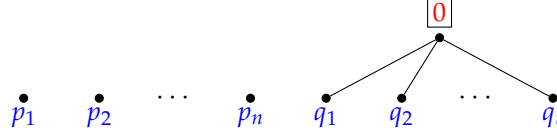


FIGURE 5. The marked poset constructed in the proof of Proposition 5.7.

Define a poset P with ground set $\{p_1, \dots, p_n, q_1, \dots, q_t, r\}$ and covering relations $q_l \prec r$ for $l = 1, \dots, t$. Define a marking on $P^* = \{r\}$ by $\lambda(r) = 0$. The marked poset obtained this way is depicted in Figure 5. Let the linear system $s(x) = b$ for $x \in \mathbb{R}^P$ be given by

$$\sum_{i=1}^n a_{ki} x_{p_i} = c_k \quad \text{for } k = 1, \dots, s,$$

$$\sum_{i=1}^n b_{li} x_{p_i} - x_{q_l} = d_l \quad \text{for } l = 1, \dots, t.$$

The conditional marked order polyhedron $\mathcal{O}(P, \lambda, s, b)$ is affinely isomorphic to Q by the map $\mathcal{O}(P, \lambda, s, b) \rightarrow Q$ sending $x \in \mathbb{R}^P$ to $(x_{p_1}, \dots, x_{p_n}) \in \mathbb{R}^n$. \square

We may now come back to the question of when a face partition π of (P, λ) still corresponds to a face of $\mathcal{O}(P, \lambda, s, b)$. As discussed at the beginning of this section, we have to decide whether $s(x) = b$ admits a solution in the relative interior of the face F'_π of $\mathcal{O}(P, \lambda)$, that is $\text{relint}(F'_\pi) \cap s^{-1}(b) \neq \emptyset$. Using the affine isomorphism induced by the quotient map this is equivalent to

$$\text{relint}(\mathcal{O}(P/\pi, \lambda/\pi)) \cap (s/\pi)^{-1}(b) \neq \emptyset.$$

Hence, we reduced the problem to deciding whether a linear system $s(x) = b$ admits a solution in the relative interior of a marked order polyhedron $\mathcal{O}(P, \lambda)$. However, even deciding whether $s(x) = b$ admits any solution in $\mathcal{O}(P, \lambda)$ is equivalent to deciding whether $\mathcal{O}(P, \lambda, s, b)$ is non-empty, which is in general just as hard as determining whether an arbitrary system of linear equations and linear inequalities admits a solution by Proposition 5.7.

We conclude that the concept of conditional marked order polyhedra is too general to obtain meaningful results. Still, in special cases the structure of an underlying poset and faces still corresponding to a subset of face partitions might be useful. An interesting class of conditional marked order polyhedra might consist of those, where P is connected and conditions are given by fixing sums along disjoint subsets of P , as is the case for Gelfand–Tsetlin polytopes with weight conditions.

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